Multi-phase mixture balance laws with phase transition

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The mixture balance laws in this project describe the dispersed flows with a large number of droplets, bubbles, or particles. Mathematically, the generic governing system can be categorized as a nonlinear resonant or weakly hyperbolic system. Until now it is still a great challenge to analyze and primarily understand these mixture balance laws even neglecting the phase transition terms. Andrianov and Warnecke in 2004 established that the Euler equations in ducts of varying cross-sectional area can be viewed as the submodel of the mixture balance laws. The main mathematical properties of the Euler equations in ducts are the same as for the generic model. For hyperbolic systems the Riemann problems serve as building blocks for the existence and uniqueness of the general solution. Hence the aim of this project was to completely solve the Riemann problem for the Euler equations in ducts. This will be used as a tool to deeply analyze and understand the generic mixture balance laws.

Models

$a_t=0,$

$$\frac{\partial aU}{\partial t}+\frac{\partial aF(U)}{\partial x}=-a'(x)H(U) \quad (1)$$

where

$U=(\rho,\rho v,\rho E)^T$

$F(U)=(\rho v,\rho v^2+p,v(\rho E+p))^T$

$H(U)=(\rho v,\rho v^2,v(\rho E+p))^T$

- a : cross—sectional area of the duct
- ρ : density
- v : velocity
- p : pressure
- $E=p/((\gamma-1)\rho)+v^2/2$

Objectives

- Completely solve (1) for the Riemann problem:

$$(a,U)=\begin{cases}(a_L,U_L), & x<0,\\ (a_R,U_R), & x>0.\end{cases}$$

- Find mechanism for nonuniqe Riemann solutions
- Single out the physical relevant solution

Cooperation

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Elementary wave curves

- The crucial point in solving Riemann problem for hyperbolic system is the construction as well as the connection of all wave curves.
- The wave curves $T_k(U_q)$, k=1,2,3 are inherited from the well studied homogeneous Euler system. In this work we assume that $T_1(u_L)\cap T_3(U_R)\neq\emptyset$.
- The 0-wave can be viewed as a 0 width transition layer located at $x=0$. It is also named the stationary wave and defined by $\partial F(U)/\partial x=-a'(x)/a(x)H(U)$.
- For given inflow state U_i , the velocity of the outflow state satisfies the following function:

$$\psi(v;U_i,a_R):=\frac{v^2}{2}+\frac{c_i^2}{\gamma-1}\left(\frac{a_L v_i}{a_R v}\right)^{\gamma-1}-\frac{v_i^2}{2}-\frac{c_i^2}{\gamma-1}.$$

The existence of the stationary wave has been first studied. It provides the methodology for the other resonant hyperbolic systems.

Lemma 1. The velocity function $\psi(v;U_i,a_R)$ decreases if the flow is subsonic; and increases if the flow is supersonic. It reaches the minimum value at the sonic state.

We denote the outflow state related to the fixed jump of the cross-sectional areas a_L and a_R and the given inflow state $U=J(a_R;U_i,a_L)$.

Theorem 2. Assume that the velocity of the stationary wave is positive, we have

1. If $a_L/a_R<1,$ the state U exists for arbitrary U_i .
2. If $a_L/a_R>1,$ the state U exists iff $m_i\leqslant\beta_l<1$ or $m_i\geqslant\beta_r$, the values β_l and β_r are the left and right solutions of

$$\varphi(m;a_L,a_R):=\frac{\gamma+1}{2(\gamma-1)}\left(\frac{a_L}{a_R}\right)^{\frac{2(\gamma-1)}{\gamma+1}}m^{\frac{\gamma-1}{\gamma+1}}-\frac{m}{2}-\frac{1}{\gamma-1}=0.$$

Lemma 3. Assume that $a_L>a_R$ and $u_i>c_i,$ define two critical duct areas:

$$a_T=a_L\left|\frac{v_i}{c_i}\right|\left[\mu^2\left(\frac{v_i}{c_i}\right)^2+1-\mu^2\right]^{-\frac{1}{2\mu^2}},\quad a_S=a_L\left|\frac{v_1^0}{c_1^0}\right|\left[\mu^2\left(\frac{v_1^0}{c_1^0}\right)^2+1-\mu^2\right]^{-\frac{1}{2\mu^2}},$$

Where $U_1^0=S_1^0(U_i)$ denotes the 0 speed 1 shock. Then we have $a_S>a_T.$

L-M and R-M curves

The L-M curve consists of 1-waves (shocks or rarefactions) and a stationary wave (not necessarily present). The R-M curves can be defined and classified in an analogous manner.

The classification of L-M curves

Case	Conditions	
I	$\mathbf{v_L-c_L<0;}$	$\mathbf{a_L>a_R}$
II	$\mathbf{v_L-c_L\leq 0;}$	$\mathbf{a_L<a_R}$
III	$\mathbf{v_L-c_L>0;}$	$\mathbf{a_L<a_R}$
IV	$\mathbf{v_L-c_L>0;}$	$\mathbf{a_L>a_R>a_S>a_T}$
V	$\mathbf{v_L-c_L>0;}$	$\mathbf{a_L>a_S>a_R>a_T}$
VI	$\mathbf{v_L-c_L>0;}$	$\mathbf{a_L>a_S>a_T>a_R}$

L—M curve for Case I

$$\begin{aligned}Q_1(U_L)&=\{U|U\in T_1(U_L),v<0\},\\ Q_2(U_L)&=\{U|U=J(a_R;U_-,a_L),\\ &\qquad U_-\in T_1(U_L),0<v<v_c\},\\ Q_3(U_L)&=\{U|U\in T_1(U_c),v>v_c\},\end{aligned}$$

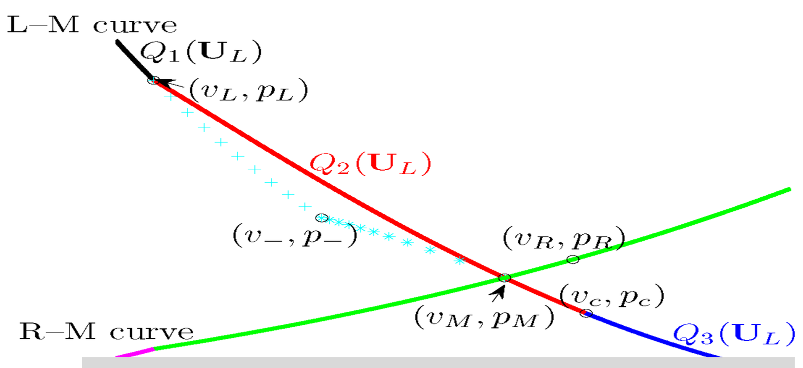


Fig 1. Example for L-M curve in case I

Example 2: L-M curve for Case IV

$$\begin{aligned}Q_1(U_L)&=\{U|U\in T_1(U_L),v<0\},\\ Q_2(U_L)&=\left\{U|U=J(a_R;U_-,a_L),U_-\in S_1^-(U_L),0<v<\hat{\bar{v}}_L\right\},\\ Q_3(U_L)&=\left\{U|U=J(a_R;U_+,a);U_+=S_1^0(U_-);U_-=J(a;U_L,a_L)\right\}\\ Q_4(U_L)&=\left\{U|U\in T_1(\bar{U}_L),v>\hat{\bar{v}}_L\right\},\end{aligned}$$

where $\hat{\bar{U}}_L=J(a_R;\hat{U}_L,a_L),\hat{U}_L=S_1^0(U_L),\hat{\bar{U}}_L=S_1^0(\bar{U}_L),\bar{U}_L=J(a_R;U_L,a_L).$

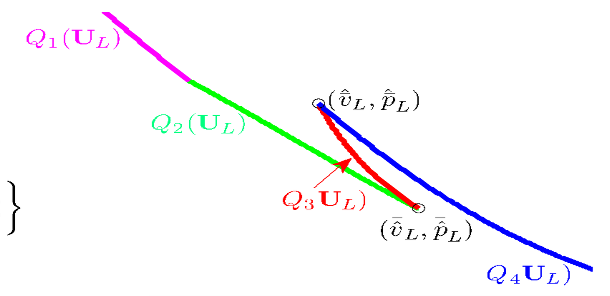
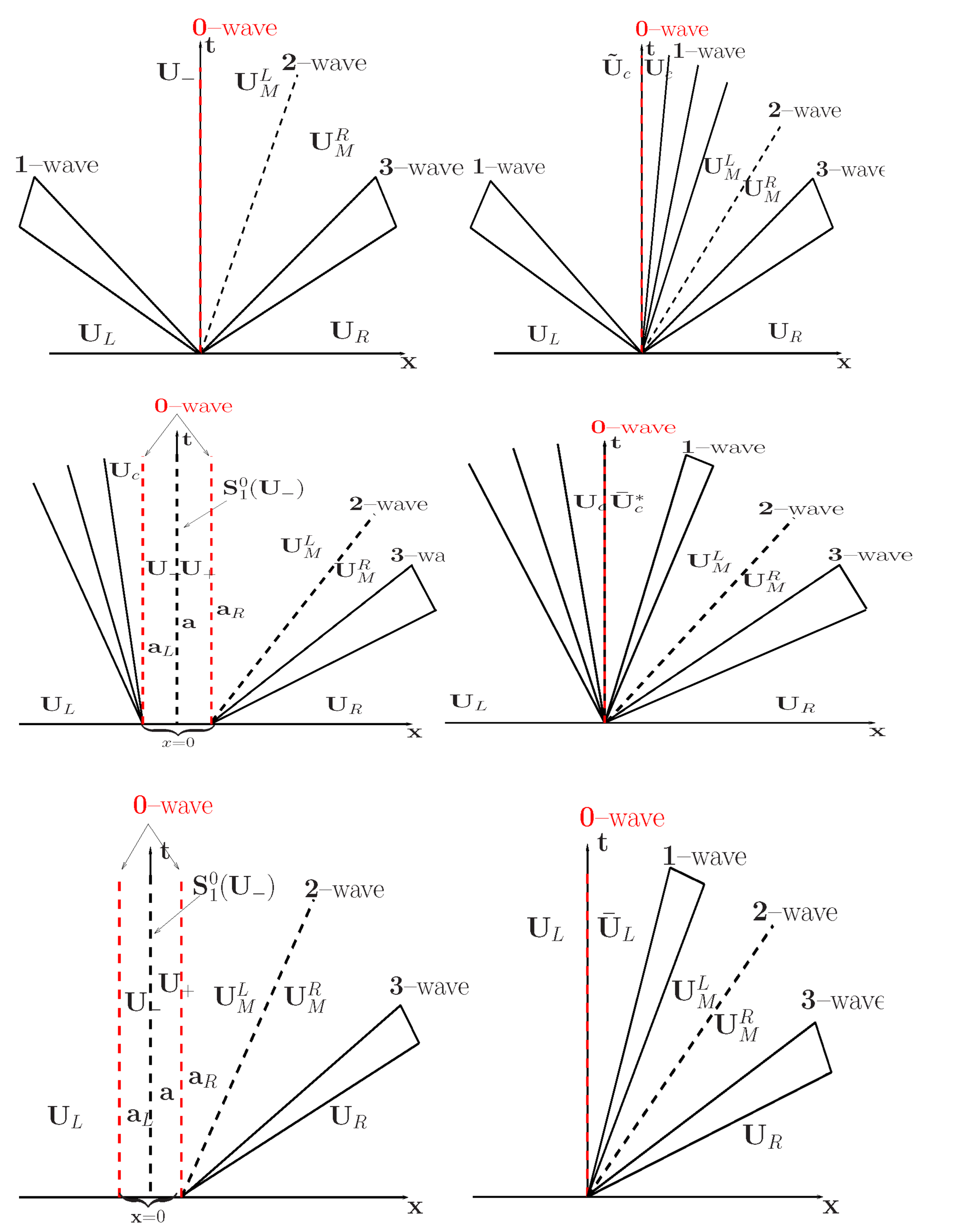


Fig 2. Bifurcation in Case IV

Results

The order of the wave configurations is: A, B, C, D, E, F



Nonunique solutions

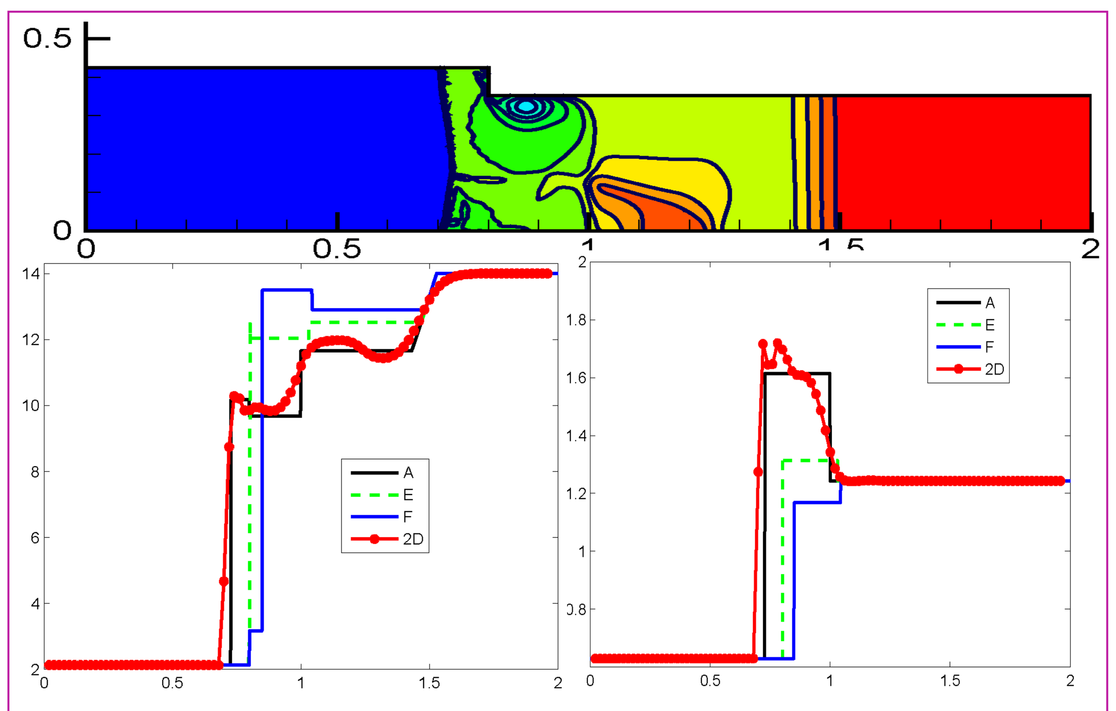


Fig 3. Top: density contour. Bottom: comparison results of density and entropy

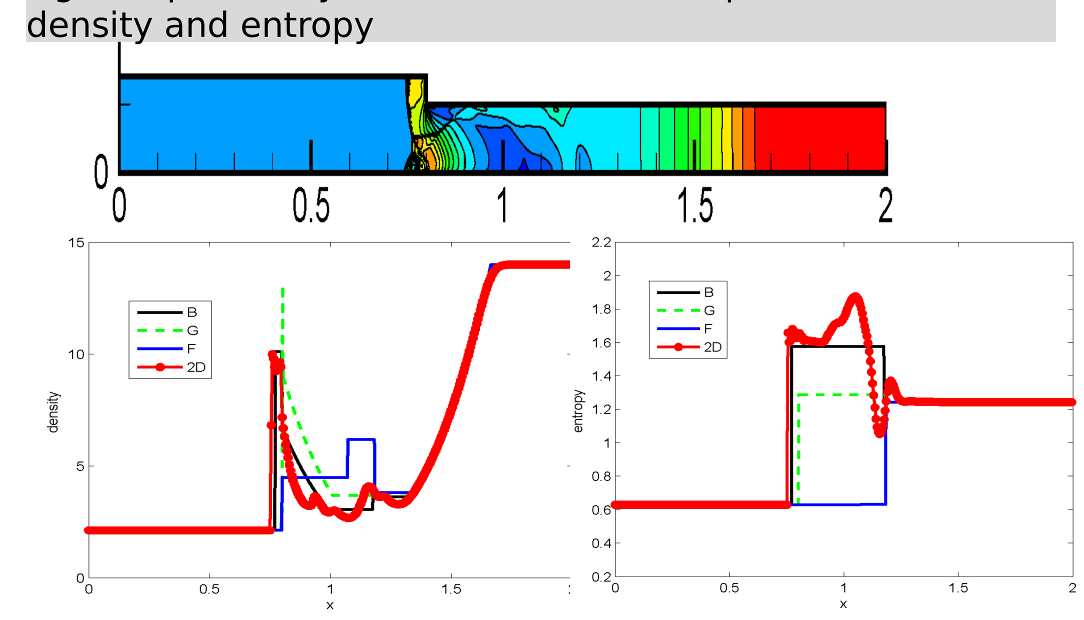


Fig 4. Top: density contour. Bottom: comparison results of density and entropy

Conclusions

- All possible exact Riemann solution to the Euler equations in a duct with discontinuous cross—sectional areas has been completely obtained by constructing the L-M and R-M curves.
- The L-M (R-M) curve is continuous and decreasing (increasing) in the (v,p) state plane for Cases I, II, and III. The Riemann solution exists uniquely in these cases.
- The L-M and R-M curves in Cases IV and V contain the bifurcation, see Fig 2. It leads to nonunique Riemann solutions. For one given initial data, there are three different solutions, see Figs. 3 and 4.
- We compare all possible non unique Riemann solutions with the numerical results of axisymmetric Euler equations by GRP scheme based on the unstructured triangle meshes. The physical relevant solution is related to the certain branches of the L-M and R-M curves

References

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